Appendix C: Methods Used for Solving Single-Digit Addition and Subtraction Problems

of the

Mathematics Framework

for California Public Schools: Kindergarten Through Grade Twelve

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Appendix C

Methods Used for Solving Single-Digit Addition and Subtraction Problems

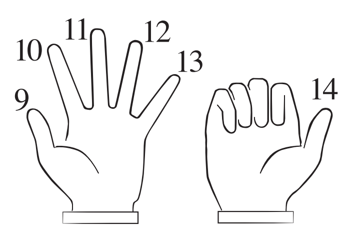
This appendix was adapted from the University of Arizona (UA) Draft K–5 Progression on Counting and Cardinality and Operations and Algebraic Thinking (UA Progressions Documents 2011a). It discusses various computational methods (levels 1, 2, and 3) that students might use to solve addition and subtraction problems. Each framework chapter for kindergarten through grade two also includes a table of “Methods Used for Solving Single-Digit Addition and Subtraction Problems” that summarizes these three methods. Additionally, the grade-level chapters provide examples and explanations of how students might use these methods to solve grade-appropriate addition and subtraction problems.

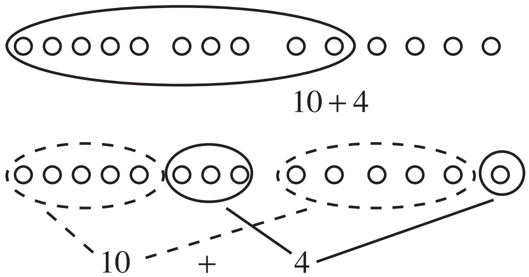
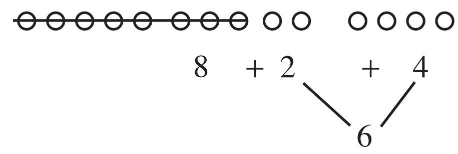
Level 1: Direct Modeling by Counting All or Taking Away

Represent the situation or numerical problem with groups of objects, a drawing, or fingers. Model the situation by composing two addend groups or decomposing a total group. Count the resulting total or addend.

- Adding ($8+6=\square$): Represent each addend by a group of objects. Put the two groups together. Count the total. Use this strategy for Add To/Result Unknown and Put Together/Total Unknown problems.

- Subtracting ($14-8=\square$): Represent the total by a group of objects. Take the known addend number of objects away. Count the resulting group of objects to find the unknown addend. Use this strategy for Take From/Result Unknown problems.

Level	$8+6=14$	$14-8=6$
Level 1: Count all	Count All a b 1 2 3 4 5 6 7 8 1 2 3 4 5 6 ○○○○○ ○○○ ○○○○○○ 1 2 3 4 5 6 7 8 9 10 11 12 13 14 c	Take Away a 1 2 3 4 5 6 7 8 9 10 11 12 13 14 ~~○○○○○ ○○○~~ ○○ ○○○○ 1 2 3 4 5 6 7 8 1 2 3 4 5 6 b
Level 2: Count on	Count On 8 ○○○○○ ○○○ ○○○○○○ 9 10 11 12 13 14 8	To solve $14-8$, I count on $8+?=14$ I took away 8  8 to 14 is 6, so $14-8=6$

Level 3: Re-compose Make a ten (general): one addend breaks apart to make 10 with the other addend. Make a ten (from 5's within each addend)	Re-compose: Make a Ten  $10 + 4$ $10 + 4$	$14 - 8$: I make a ten for $8 + ? = 14$  $8 + 2 + 4$ 6 $8 + 6 = 14$
Doubles $= n$	$6 + 8$ $= 6 + 6 + 2$ $= 12 + 2 = 14$	

Note: Many children attempt to count down for subtraction, but counting down is difficult and error-prone. Children are much more successful with counting on; it makes subtraction as easy as addition.

Level 2: Counting On

Embed an addend within the total (the addend is perceived simultaneously as an addend and as part of the total). Count this total, but abbreviate the counting by omitting the count of this addend; instead, begin with the number word of this addend. The count is tracked and monitored in some way (e.g., with fingers, objects, mental images of objects, body motions, or other count words).

For addition, the count is stopped when the amount of the remaining addend has been counted. The last number word is the total. For subtraction, the count is stopped when the total occurs in the count. The tracking method indicates the difference (seen as an unknown addend).

Counting on may be used to find the total or to find an addend. These look the same to an observer. The difference is what is monitored: the total or the known addend. Some students count down to solve subtraction problems, but this method is less accurate and more difficult than counting on. Counting on is not a rote method. It requires several connections between cardinal and counting meanings of the number words, as well as extended experience with Level 1 methods in kindergarten.

- Adding (e.g., $8 + 6 = \square$) uses counting on to find a total: One counts on from the first addend (or the larger number is taken as the first addend). Counting on is monitored so that it stops when the second addend has been counted on. The last number word is the total.
- Finding an unknown addend (e.g., $8 + \square = 14$): One counts on from the known addend. The "keeping track" method is monitored so that counting on stops when the known total has been reached. The "keeping track" method tells the unknown addend.
- Subtracting ($14 - 8 = \square$): One thinks of subtracting as finding the unknown addend, as $8 + \square = 14$, and uses counting on to find an unknown addend (as above).

In the Glossary of this framework, table GL-4 includes problems that can be solved with Level 1 methods in kindergarten or by using Level 2 methods: counting on to find the total (adding) or counting on to find the unknown addend (subtracting). Level 2 and 3 methods are generally used in grades one and two.

Finding an unknown addend (e.g., $8+\square=14$) is used for Add To/Change Unknown problems, Put Together/Take Apart/Addend Unknown problems, and Compare/Difference Unknown problems. It is also used for Take From/Change Unknown ($14-\square=8$) problems after a student has decomposed the total into two addends, which means they can represent the situation as $14-8=\square$.

Adding or subtracting by counting on is used by some students for each of the kinds of Compare problems (see the equations in table GL-4 of the Glossary). Students in grade one do not necessarily master the Compare Bigger Unknown or Smaller Unknown problems that use misleading language (such as the words *fewer* or *more than*). These problem types appear in the bottom row of table GL-4 of the Glossary.

Solving an equation such as $6+8=\square$ by counting on from 8 relies on the understanding that $8+6$ gives the same total—an implicit use of the commutative property without the accompanying written representation $6+8=8+6$.

Level 3: Convert to an Easier Equivalent Problem

Decompose an addend and compose a part with another addend.

The following methods can be used to add or to find an unknown addend (and thus to subtract). The methods implicitly use the associative property.

Adding

Make a ten. For example, for $8+6=\square$,

$$8+\underline{6}=8+\underline{2+4}=10+4=14,$$

$$\text{so } 8+6 \text{ becomes } 10+4.$$

Doubles plus or minus 1. For example, for $6+7=\square$,

$$6+\underline{7}=6+\underline{6+1}=12+1=13,$$

$$\text{so } 6+7 \text{ becomes } 12+1.$$

Finding an unknown addend

Make a ten. For example, for $8+\square=14$,

$$8+\underline{2}=10 \text{ and } \underline{4} \text{ more makes } 14$$

$$\underline{2+4}=6$$

So $8+\square=14$ is done as two steps: how many up to 10 and how many over 10 (which can be seen in the ones place of 14).

Doubles plus or minus 1. For example, for $6+\square=13$,

$$6+\underline{6+1}=12+1$$
$$\underline{6+1}=7$$

So $6+\square=13$ is done as two steps: how many up to $12(6+6)$ and how many from 12 to 13.

Subtracting

Thinking of subtracting as finding an unknown addend.

For example, using the methods shown above, solve $14-8=\square$ or $13-6=\square$ as $8+\square=14$ or $6+\square=13$ (make a ten or use doubles plus or minus 1).

Thinking of subtraction as "Make a ten first" or "Breaking down to 10"

For example, $15-8=\square$ can be done in two steps:

$$15-8=(\underline{15-5})-3=\underline{10}-3=7$$

Students think how to make a $10(15-5)$ and then subtract what remains from the subtrahend (the number being subtracted—3 in the example).

The Level 1 and Level 2 problem types can be solved by using these Level 3 methods.

Level 3 problem types can be solved by representing the situation with an equation or drawing, then re-representing to create a situation solved by adding, subtracting, or finding an unknown addend (as shown above) by methods at any level, but usually at Level 2 or 3. Many students show in their writing only part of this multi-step process of re-representing the situation.

- Students re-represent Add To/Start Unknown situations by using the commutative property (formally or informally). For example, $\square+6=14$ is re-represented as $6+\square=14$.
- Students re-represent Take From/Start Unknown situations by reversing them. For example, $\square-8=6$ is re-represented as $6+8=\square$, which may then be solved by counting on from 8 or using a Level 3 method.

At Level 3, the Compare problems with misleading language can be solved by representing the known quantities in a diagram that shows the bigger quantity in relation to the smaller quantity. The diagram allows the student to find a correct situation by representing the difference between quantities and seeing the relationship among the three quantities. Such diagrams are the same as those used for the other versions of Compare situations; focusing on which quantity is bigger and which is smaller helps students to overcome the misleading language.

Some students may solve Level 3 problem types by re representing (as described above), but use Level 2 counting on.

As students move through levels of solution methods, they increasingly use equations to represent problem situations as situation equations and then to re-represent the situation with a solution equation or a solution computation. They relate equations to diagrams, which facilitates the process of re-representing. Labeling diagrams may help connect the parts of the diagram to the corresponding parts of the situation. However, students may know and understand things that they may not use for a given solution of a problem, as they increasingly do various representing and re-representing steps mentally.

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