Instructional Strategies Chapter

of the

Mathematics Framework

for California Public Schools:
Kindergarten Through Grade Twelve

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Instructional Strategies

This chapter is intended to enhance teachers' repertoire, not prescribe the use of any particular instructional strategy. For any given instructional goal, teachers may choose among a wide range of instructional strategies, and effective teachers look for a fit between the material to be taught and strategies for teaching that material. (See the grade-level and course-level chapters for more specific examples.) Ultimately, teachers and administrators must decide which instructional strategies are most effective in addressing the unique needs of individual students.

In a standards-based curriculum, effective lessons, units, or modules are carefully developed and are designed to engage all members of the class in learning activities that aim to build student mastery of specific standards. Such lessons typically last at least 50 to 60 minutes daily (excluding homework). The goal that all students should be ready for college and careers by mastering the standards is central to the California Common Core State Standards for Mathematics (CA CCSSM) and this mathematics framework. Lessons need to be designed so that students are regularly exposed to new information while building conceptual understanding, practicing skills, and reinforcing their mastery of previously introduced information. The teaching of mathematics must be carefully sequenced and organized to ensure that all standards are taught at some point and that prerequisite skills form the foundation for more advanced learning. However, teaching should not proceed in a strictly linear order, requiring students to master each standard completely before they are introduced to another. Practice that leads toward mastery can be embedded in new and challenging problems that promote conceptual understanding and fluency in mathematics.

Before instructional strategies available to teachers are discussed, three important topics for the CA CCSSM will be addressed: Key Instructional Shifts, Standards for Mathematical Practice, and Critical Areas of Instruction at each grade level.

Key Instructional Shifts

Understanding how the CA CCSSM differ from previous standards—and the necessary shifts called for by the CA CCSSM—is essential to implementing California's newest mathematics standards. The three key shifts or principles on which the standards are based are *focus*, *coherence*, and *rigor*. Teachers, schools, and districts should concentrate on these three principles as they develop a common understanding of best practices and move forward with the implementation of the CA CCSSM.

Each grade-level chapter of the framework begins with the following summary of the principles.

Standards for Mathematical Content

The Standards for Mathematical Content emphasize key content, skills, and practices at each grade level and support three major principles:

- **Focus**—Instruction is focused on grade-level standards.
- **Coherence**—Instruction should be attentive to learning across grades and to linking major topics within grades.
- **Rigor**—Instruction should develop conceptual understanding, procedural skill and fluency, and application.

Focus requires that the scope of content in each grade, from kindergarten through grade twelve, be significantly narrowed so that students experience more deeply the remaining content. Surveys suggest that postsecondary instructors value greater mastery of prerequisites over shallow exposure to a wide array of topics with dubious relevance to postsecondary work.

Coherence is about math making sense. When people talk about coherence, they often talk about making connections between topics. The most important connections are vertical: the links from one grade to the next that allow students to progress in their mathematical education. That is why it is critical to think across grades and examine the progressions in the standards to see how major content develops over time.

Rigor has three aspects: *conceptual understanding*, *procedural skill and fluency*, and *application*. Educators need to pursue, with equal intensity, all three aspects of rigor in the major work of each grade.

- The word *understand* is used in the standards to set explicit expectations for conceptual understanding. The word *fluently* is used to set explicit expectations for fluency.
- The phrase *real-world problems* (and the star [★] symbol) are used to set expectations and indicate opportunities for applications and modeling.

The three aspects of rigor are critical to day-to-day and long-term instructional goals for teachers. Because of this importance, they are described further below:

- *Conceptual understanding.* Teachers need to teach more than how to "get the right answer," and instead should support students' ability to acquire

Rigor in the Curricular Materials

"To date, curricula have not always been balanced in their approach to these three aspects of rigor. Some curricula stress fluency in computation without acknowledging the role of conceptual understanding in attaining fluency and making algorithms more learnable. Some stress conceptual understanding without acknowledging that fluency requires separate classroom work of a different nature. Some stress pure mathematics without acknowledging that applications can be highly motivating for students and that a mathematical education should make students fit for more than just their next mathematics course. At another extreme, some curricula focus on applications, without acknowledging that math doesn't teach itself. The standards do not take sides in these ways, but rather they set high expectations for all three components of rigor in the major work of each grade. Of course, that makes it necessary that we focus—otherwise we are asking teachers and students to do more with less."

—National Governors Association Center for Best Practices, Council of Chief State School Officers (NGA/CCSSO) 2013, 4

concepts from several perspectives so that students are able to see mathematics as more than a set of mnemonics or discrete procedures. Students demonstrate solid conceptual understanding of core mathematical concepts by applying these concepts to new situations as well as writing and speaking about their understanding. When students learn mathematics conceptually, they understand *why* procedures and algorithms work, and doing mathematics becomes meaningful because it makes sense.

- *Procedural skill and fluency.* Conceptual understanding is not the only goal; teachers must also structure class time and homework time for students to practice procedural skills. Students develop fluency in core areas such as addition, subtraction, multiplication, and division so that they are able to understand and manipulate more complex concepts. Note that fluency is *not* memorization without understanding; it is the outcome of a carefully laid-out learning progression that requires planning and practice.
- *Application.* The CA CCSSM require application of mathematical concepts and procedures throughout all grade levels. Students are expected to use mathematics and choose the appropriate concepts for application even when they are not prompted to do so. Teachers should provide opportunities in all grade levels for students to apply mathematical concepts in real-world situations, as this motivates students to learn mathematics and enables them to transfer their mathematical knowledge into their daily lives and future careers. Teachers in content areas outside mathematics (particularly science) ensure that students use grade-level-appropriate mathematics to make meaning of and access content.

These three aspects of rigor should be taught in a balanced way. Over the years, many people have taken sides in a perceived struggle between teaching for conceptual understanding and teaching procedural skill and fluency. The CA CCSSM present a balanced approach: teaching *both*, understanding that each informs the other. Application helps make mathematics relevant to the world and meaningful for students, enabling them to maintain a productive disposition toward the subject so as to stay engaged in their own learning.

Throughout this chapter, attention will be paid to the three major instructional shifts (or principles). Readers should keep in mind that many of the standards were developed according to findings from research on student learning (e.g., on students' [in kindergarten through grade five] understanding of the four operations or on the learning of standard algorithms in grades two through six). The task for teachers, then, is to develop the most effective means for teaching the content of the CA CCSSM to diverse student populations while staying true to the intent of the standards.

Standards for Mathematical Practice

The Standards for Mathematical Practice (MP) describe expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" of longstanding importance in mathematics education. The first of these are the National Council of Teachers of Mathematics process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning; strategic compe-

tence; conceptual understanding (comprehension of mathematical concepts, operations, and relations); procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently, and appropriately); and productive disposition, which is the habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (NGA/CCSSO 2010q, 6).

Instruction must be designed to incorporate these standards effectively. Teachers should analyze their curriculum and identify where content and practice standards intersect. The grade-level chapters of this framework contain some examples where connections between the MP standards and the Standards for Mathematical Content are identified. Teachers should be aware that it is not possible to address every MP standard in every lesson and that, conversely, because the MP standards are themselves interconnected, it would be difficult to address only a single MP standard in a given lesson.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

The MP standards establish certain behaviors of mathematical expertise, sometimes referred to as "habits of mind" that *should be explicitly taught*. For example, students in third grade are not expected to know from the outset what a viable argument would look like (**MP.3**); the teacher and other students set the expectation level by critiquing reasoning presented to the class. The teacher is also responsible for creating a safe atmosphere in which students can engage in mathematical discourse that comes with rich tasks. Likewise, students in higher mathematics courses realize that the level of mathematical argument has increased: they use appropriate language and logical connections to construct and explain their arguments and communicate their reasoning clearly and effectively. The teacher serves as the guide in developing these skills. Later in this chapter, mathematical tasks are presented that exemplify the intersection of the mathematical practice and content standards.

Critical Areas of Instruction

At the beginning of each grade-level chapter in this framework, a brief summary of the Critical Areas of Instruction for the grade at hand is presented. For example, the following summary appears in the chapter on grade five:

> In grade five, instructional time should focus on three critical areas: (1) developing fluency with addition and subtraction of fractions and developing understanding of the multiplication of fractions and of division of fractions in limited cases (unit fractions divided by whole numbers and whole numbers divided by unit fractions); (2) extending division to two-digit divisors, integrating decimal fractions into the place-value system, developing understanding of operations with decimals to hundredths, and developing fluency with whole-number and decimal operations; and (3) developing understanding of volume (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2010l). Students also fluently multiply multi-digit whole numbers using the standard algorithm.

The Critical Areas of Instruction should be considered examples of expectations of focus, coherence, and rigor for each grade level. The following points refer to the critical areas in grade five:

Please see the CA CCSSM publication (CDE 2013a) for further explanation of these Critical Areas for each grade level. The publication is available at http://www.cde.ca.gov/re/cc/ (accessed September 1, 2015).

- Critical Area (1) refers to students using their understanding of equivalent fractions and fraction models to develop fluency with fraction addition and subtraction. Clearly, this is a major *focus* of the grade.
- Critical Area (1) is connected to Critical Area (2), as students relate their understanding of decimals as fractions to making sense out of rules for multiplying and dividing decimals, illustrating *coherence* at this grade level.
- A vertical example (i.e., one that spans grade levels) of *coherence* is evident by noticing that students have performed addition and subtraction with fractions with like denominators in grade four and reasoned about equivalent fractions in that grade; they further their understanding to add and subtract all types of fractions in grade five.
- Finally, there are several examples of *rigor* in grade five: in Critical Area (1), students apply their *understanding* of fractions and fraction models; also in Critical Area (1), students *develop fluency* in calculating sums and differences of fractions; and in Critical Area (3), students solve *real-world problems* that involve determining volumes.

These are just a few examples of focus, coherence, and rigor from the Critical Areas of Instruction in grade five. Critical Areas of Instruction, which should be viewed by teachers as a reference for planning instruction, are listed at the beginning of each grade-level chapter. Additional examples of focus, coherence, and rigor appear throughout the grade-level chapters, and each grade-level chapter includes a table that highlights the content emphases at the cluster level for the grade-level standards. The bulk of instructional time should be given to "Major" clusters and the standards that are listed with them.

General Instructional Models

Teachers are presented with the task of effectively delivering instruction that is aligned with the CA CCSSM and pays attention to the Key Instructional Shifts, the Standards for Mathematical Practice, and the Critical Areas of Instruction at each grade level (i.e., instructional features). This section describes several general instructional models. Each model has particular strengths related to the aforementioned instructional features. Although classroom teachers are ultimately responsible for delivering instruction, research on how students learn in classroom settings can provide useful information to both teachers and developers of instructional resources.

Because of the diversity of students in California classrooms and the new demands of the CA CCSSM, a combination of instructional models and strategies will need to be considered to optimize student learning. Cooper (2006, 190) lists four overarching principles of instructional design for students to achieve learning with understanding:

1. Instruction is organized around the solution of meaningful problems.
2. Instruction provides scaffolds for achieving meaningful learning.

3. Instruction provides opportunities for ongoing assessment, practice with feedback, revision, and reflection.
4. The social arrangements of instruction promote collaboration, distributed expertise, and independent learning.

Mercer and Mercer (2005) suggest that instructional models may range from explicit to implicit instruction:

Explicit Instruction	Interactive Instruction	Implicit Instruction
Teacher serves as the provider of knowledge	Instruction includes both explicit and implicit methods	Teacher facilitates student learning by creating situations in which students discover new knowledge and construct their own meanings
Much direct teacher assistance	Balance between direct and non-direct teacher assistance	Non-direct teacher assistance
Teacher regulation of learning	Shared regulation of learning	Student regulation of learning
Directed discovery	Guided discovery	Self-discovery
Direct instruction	Strategic instruction	Self-regulated instruction
Task analysis	Balance between part-to-whole and whole-to-part	Unit approach
Behavioral	Cognitive/metacognitive	Holistic

Mercer and Mercer further suggest that the type of instructional models to be used during a lesson will depend on the learning needs of students and the mathematical content presented. For example, explicit instruction models may support practice to mastery, the teaching of skills, and the development of skills and procedural knowledge. On the other hand, implicit models link information to students' background knowledge, developing conceptual understanding and problem-solving abilities.

5E Model

Carr et al. (2009) link the 5E (interactive) model to three stages of mathematics instruction: introduce, investigate, and summarize. As its name implies, this model is based on a recursive cycle of five cognitive stages in inquiry-based learning: (a) engage, (b) explore, (c) explain, (d) elaborate, and (e) evaluate. Teachers have a multi-faceted role in this model. As a facilitator, the teacher nurtures creative thinking, problem solving, interaction, communication, and discovery. As a model, the teacher initiates thinking processes, inspires positive attitudes toward learning, motivates, and demonstrates skill-building techniques. Finally, as a guide, the teacher helps to bridge language gaps and foster individuality, collaboration, and personal growth. The teacher flows in and out of these various roles within each lesson.

The Three-Phase Model

The three-phase (explicit) model represents a highly structured and sequential strategy utilized in direct instruction. It has proved to be effective for teaching information and basic skills during whole-class instruction. In the first phase, the teacher introduces, demonstrates, or explains the new concept or strategy, asks questions, and checks for understanding. The second phase is an intermediate step designed to result in the independent application of the new concept or described strategy. When the teacher is satisfied that the students have mastered the concept or strategy, the third phase is implemented: students work independently and receive opportunities for closure. This phase also often serves, in part, as an assessment of the extent to which students understand what they are learning and how they use their knowledge or skills in the larger scheme of mathematics.

Singapore Math

Singapore math (an interactive instructional approach) emphasizes the development of strong number sense, excellent mental-math skills, and a deep understanding of place value. It is based on Bruner's (1956) principles, a progression from concrete experience using manipulatives, to a pictorial stage, and finally to the abstract level or algorithm. This sequence gives students a solid understanding of basic mathematical concepts and relationships before they start working at the abstract level. Concepts are taught to mastery, then later revisited but not retaught. The Singapore approach focuses on the development of students' problem-solving abilities. There is a strong emphasis on *model drawing*, a visual approach to solving word problems that helps students organize information and solve problems in a step-by-step manner. For additional information on Singapore math, please visit the National Center for Education Statistics Web site (https://nces.ed.gov/pubsearch/pubsinfo.asp?pubid=WWCIRMSSM09 [accessed June 25, 2015]).

Concept Attainment Model

Concept attainment is an interactive, inductive model of teaching and learning that asks students to categorize ideas or objects according to critical attributes. During the lesson, teachers provide examples and non-examples, and then ask students to (1) develop and test hypotheses about the exemplars, and (2) analyze the thinking processes that were utilized. To illustrate, students may be asked to categorize polygons and non-polygons in a way that is based upon a pre-selected definition. Through concept attainment, the teacher is in control of the lesson by selecting, defining, and analyzing the concept beforehand and then encouraging student participation through discussion and interaction. This strategy may be used to introduce, strengthen, or review concepts, and as formative assessment (Charles and Senter 2012).

The Cooperative Learning Model

An important component of the mathematical practice standards is having students work together to solve problems. Students actively engage in providing input and assess their efforts in learning the content. They construct viable arguments, communicate their reasoning, and critique the reasoning of others (**MP.3**). The role of the teacher is to guide students toward desired learning outcomes. The cooperative learning model is an example of implicit instruction and involves students working either

as partners or in mixed-ability groups to complete specific tasks. It assists teachers in addressing the needs of diverse student populations, which are common in California's classrooms. The teacher presents the group with a problem or a task and sets up the student activities. While the students work together to complete the task, the teacher monitors progress and assists student groups when necessary (Charles and Senter 2012; Burden and Byrd 2010).

Cognitively Guided Instruction

The cognitively guided (implicit) instruction model calls for the teacher to have students consider different ways to solve a problem. A variety of student-generated strategies are used to solve a particular problem—for example, using plastic cubes to model the problem, counting on fingers, and using knowledge of number facts to figure out the answer. The teacher then asks the students to explain their reasoning process. They share their explanations with the class. The teacher may also ask the students to compare different strategies. Students are expected to explain and justify their strategies and, along with the teacher, take responsibility for deciding whether a strategy that is presented is viable.

This instructional model puts more responsibility on the students. Rather than being asked to simply apply a formula to virtually identical mathematics problems, students are challenged to use reasoning that makes sense to them in solving the problem and to find their own solutions. In addition, students are expected to publicly explain and justify their reasoning to their classmates and the teacher. Finally, teachers are required to open their instruction to students' original ideas and to guide each student according to his or her own developmental level and way of reasoning.

Expecting students to solve problems using mathematical reasoning and sense-making and then explain and justify their thinking has a major impact on students' learning. For example, students who develop their own strategies to solve addition problems are likely to intuitively use the commutative and associative properties of addition in their strategies. When students use their own strategies to solve problems and then justify these strategies, this contributes to a positive disposition toward learning mathematics (Wisconsin Center for Education Research 2007; National Center for Improving Student Learning and Achievement in Mathematics and Science 2000).

Problem-Based Learning

The MP standards emphasize the importance of making sense of problems and persevering in solving them (**MP.1**), reasoning abstractly and quantitatively (**MP.2**), and solving problems that are based upon "everyday life, society, and the workplace" (**MP.4**). Implicit instruction models, such as problem-based (interactive) learning, project-based learning, and inquiry-based learning, provide students with the time and support to successfully engage in mathematical inquiry by collecting data and testing hypotheses. Burden and Byrd (2010) attribute John Dewey's model of reflective thinking as the basis of the instructional model: "(a) Identify and clarify a problem; (b) Form hypotheses; (c) Collect data; (d) Analyze and interpret the data to test the hypotheses; and (e) Draw conclusions" (Burden and Byrd 2010, 145). These researchers suggest two approaches for problem-based learning: guided and unguided inquiry. During *guided inquiry*, the teacher provides the data and then questions the students so that they can arrive at a solution. Through *unguided inquiry*, students take responsibility for analyzing data and coming to conclusions.

In problem-based learning, students work either individually or in cooperative groups to solve challenging problems with real-world applications. The teacher poses the problem or question, assists when necessary, and monitors progress. Through problem-based activities, "students learn to think for themselves and show resourcefulness and creativity" (Charles and Senter 2012, 125). Martinez (2010, 149) cautions that when students engage in problem solving, they must be allowed to make mistakes: "If teachers want to promote problem solving, they need to create a classroom atmosphere that recognizes errors and uncertainties as inevitable accoutrements of problem solving." Through class discussion and feedback, student errors become the basis of furthering understanding and learning (Ashlock 1998). (For additional information, refer to appendix B [Mathematical Modeling].)

This is just a sampling of instructional models that have been researched across the globe. Ultimately, teachers and administrators must determine what works best for their student populations. Teachers may find that a combination of several instructional approaches is appropriate.

Strategies for Mathematics Instruction

As teachers progress through their careers, they develop a repertoire of instructional strategies. This section discusses several instructional strategies for mathematics instruction, but it is certainly not an exhaustive list. Teachers are encouraged to seek other mathematics teachers, professional learning from county offices of education, the California Mathematics Project, other mathematics education professionals, and Internet resources to continue building their repertoire.

Discourse in Mathematics Instruction

The MP standards call for students to make sense of problems (**MP.1**), construct viable arguments (**MP.3**), and model with mathematics (**MP.4**). Students are expected to communicate their understanding of mathematical concepts, receive feedback, and progress to deeper understanding. Ashlock (1998, 66) concludes that when students communicate their mathematical learning through discussions and writing, they are able to "relate the everyday language of their world to math language and to math symbols." Van de Walle (2007, 86) adds that the process of writing enhances the thinking process by requiring students to collect and organize their ideas. Furthermore, as an assessment tool, student writing "provides a unique window to students' thoughts and the way a student is thinking about an idea."

Number/Math Talks (Mental Math). Parrish (2010) describes number talks as:

> classroom conversations around purposefully crafted computation problems that are solved mentally. The problems in a number talk are designed to elicit specific strategies that focus on number relationships and number theory. Students are given problems in either a whole- or small-group setting and are expected to mentally solve them accurately, efficiently, and flexibly. By sharing and defending their solutions and strategies, students have the opportunity to collectively reason about numbers while building connections to key conceptual ideas in mathematics. A typical classroom number talk can be conducted in five to fifteen minutes. (Parrish 2010, xviii)

During a number talk, the teacher writes a problem on the board and gives students time to solve the problem mentally. Once students have found an answer, they are encouraged to continue finding efficient strategies while others are thinking. They indicate that they have found other approaches by raising another finger for each solution. This quiet form of acknowledgment allows time for students

to think, while the process continues to challenge those who already have an answer. When most of the students have indicated they have a solution and a strategy, the teacher calls for answers. All answers—correct and incorrect—are recorded on the board for students to consider.

Next, the teacher asks a student to defend her answer. The student explains her strategy, and the teacher records the student's thinking on the board exactly as the student explains it. The teacher serves as the facilitator, questioner, listener, and learner. The teacher then has another student share a different strategy and records his thinking on the board. The teacher is not the ultimate authority, but allows the students to have a "sense of shared authority in determining whether an answer is accurate" (Parrish 2010, 11).

Here are a few questions that teachers can ask:

- How did you solve this problem?
- How did you get your answer?
- How is one strategy similar to or different from another strategy?

Five Practices for Orchestrating Productive Mathematics Discussions. Smith and Stein (2011) identify five practices that assist teachers in facilitating instruction that advances the mathematical understanding of the class:

- Anticipating
- Monitoring
- Selecting
- Sequencing
- Connecting

Organizing and facilitating productive mathematics discussions for the classroom take a great deal of preparation and planning. Prior to giving a task to students, the teacher should anticipate the likely responses that students will have so that they are prepared to facilitate the lesson. Students usually come up with a variety of strategies, but it is helpful if teachers have already anticipated some of the strategies when leading the discussion. The teacher then poses the problem and gives the task to the students. The teacher monitors the responses while students work individually, in pairs, or in small groups. The teacher pays attention to the different strategies that students use. To conduct the "share and summarize" portion of the lesson, the teacher selects a student to present his or her mathematical work and sequences the sharing so that the various strategies are presented in a specific order, to highlight the mathematical goal of the lesson. As the teacher conducts the discussion, he or she deliberately asks questions to connect responses to the key mathematical ideas.

Student Engagement Strategies

Building a list of robust student engagement strategies is essential for all teachers. When students are engaged in the classroom, they remain focused and on task. Good classroom management and effective teaching and learning result from student engagement. The table below, provided by the Rialto Unified School District, illustrates several student engagement strategies for the mathematics classroom.

Student Engagement Strategy	Description	Math Example
Appointment Clock	Students partner to make appointments for discussions or work (a good grouping strategy).	Students are given a page with a clock printed on it. They use the clock to set appointments with other students to discuss math problems.
Carousel-Museum Walk	Each group posts sample work on the wall, and the leader for that group stands near the work while the rest of the group circulates around the room, looking at all the samples.	Each group is given a poster board and math problem to work on. When all groups have finished their work, each poster is affixed to the classroom walls. Each leader stays close to the poster created by his or her group and explains the work, while the other students walk around the room looking at other groups' work.
Charades	Students act out a scenario, individually or with a team.	Students work in teams to act out word problems while others try to solve the problems.
Clues (Barrier Games)	One partner has a picture of information that the other student does not have. Sitting back to back or using a visual barrier, students communicate to complete the task.	Working in pairs, each student communicates a different problem to the other student, who has to try to solve the problem from the information provided by the first student. The students sit with a barrier between them during the activity.
Coming to Consensus	Sharing their individual ideas, the group comes to a consensus and reveals that consensus to the entire class.	Each member of the group shares an answer to a given problem, the steps used, and so forth. When the group comes to a consensus, they reveal it to the entire class.
Explorers and Settlers	Assign half the class to be explorers and half to be settlers. Explorers seek a settler to discuss a question. Students may exchange roles and repeat the process.	Half of the students are designated as explorers who have a math term or problem. The other students are designated as settlers who have the definitions or answers. Explorers seek the settler with the correct answers and discuss the information.

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Student Engagement Strategy	Description	Math Example
Find My Rule	Students are given cards and must find the person who matches their card. One person has a card with a rule, and the other has an example of that rule. This is a great strategy for practicing inductive/deductive reasoning. It also works well for grouping students randomly and developing problem-solving skills.	Two types of cards are prepared: one with a problem and the other with the rule pertaining to that problem. Students circulate throughout the room to match the cards that are connected or related to the rule. Once all members of the group have been found, group members articulate the rule and how the group is connected.
Find Your Partner	Each student is given a card that matches another student's card in some way.	Example cards: Rectangle:  Prime Number: 37 Problem $9+7$: Solution 16 Polynomial with degree 3: $4x^3-3x^2+5x-7$
Four Corners	Assign each corner of the room a category related to a topic. Students write which category they are most interested in, giving reasons, and then form groups in those corners. The activity could be adapted for different levels.	The corners of the room are numbered 2, 3, 4, and 5. Students are divided into four groups, and each group is sent to a corner. The teacher then poses a problem whose answer is a multiple of 2, 3, 4, or 5. Students in a corner that is a factor of that number will move to a different corner. If the teacher calls out 6, students in the corners labeled 2 and 3 will move. The activity ends with a prime-number answer, and students return to their seats.
Give One, Get One	After brainstorming ideas, students circulate among other students, giving one idea and receiving one. Students fold a piece of paper lengthwise to label the left side "Give one" and the right side "Get one."	The teacher gives the class a multi-step problem to solve within a specific time limit. On the "Give one" side of the paper, students name all the steps they know before finding a partner. Partner A gives an answer to partner B. If partner B has that answer, both students check it off. If partner B does not have the answer, partner B writes it on the "Get one" side. Students repeat the process with partner B going first. Once both partners have exchanged ideas, they raise their hands, find new partners, and continue until the teacher says to stop.

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Student Engagement Strategy	Description	Math Example
Inside/Outside Circle	Students stand or sit in two concentric circles, creating partners who face one another. The teacher poses a question to the class, and one partner responds. At a signal, the inner circle or outer circle rotates, and the conversation continues.	Students share information to solve problems. The teacher (or student) prepares question cards for each student. The inner-circle students ask a question from their card, the outer-circle students answer, and then these partners discuss the problem before switching roles. Once both students have asked and answered a question, the inner circle rotates clockwise to a new partner.
Jigsaw	A group of students is assigned a portion of text; these students then teach that portion to the remainder of the class.	"Factoring Jigsaw" is a game in which each student becomes an expert on a different concept or procedure in the factoring process and then teaches that concept to other students.
KWL	A cognitive graphic organizer sets the stage for learning. The teacher asks students to identify what they already **K**now, what they **W**ant to know, and what they need to do to **L**earn the skill or concept.	Math teachers use KWL as a diagnostic tool to determine student readiness, using pretest questions and a KWL chart.
Line Up (class building)	Students line up in a particular order given by the teacher (e.g., alphabetically by first name, by birth date, shortest to tallest, and so on).	Students line up in order by the number given to them: square root, fraction, decimal, or multiples of a given number. Once in line, they explain how they found their place. This is a good activity for the first day of class.
Making a List	Two students, using one word or phrase, add items to a list.	Students receive a multi-step or word problem and name the steps needed to solve the problem.

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Student Engagement Strategy	Description	Math Example
Numbered Heads Together	This is a cooperative learning strategy that holds each student accountable for learning the material. Students are placed in groups, and each person is given a number (from one to the maximum number in each group). The teacher poses a question, and students "put their heads together" to figure out the answer. The teacher calls a specific number to respond as spokesperson for the group. With students working together in a group, this strategy ensures that each member knows the answer to problems or questions asked by the teacher. Because no one knows which number will be called, all team members must be prepared.	Each group is given a problem to solve. The student whose number is called explains how the group came up with their answer.
Quiz, Quiz, Trade	Using two-sided cards prepared in advance by the teacher, students in pairs quiz each other, trade cards, and then find another partner.	May be used to help students review math vocabulary, discuss math facts, or improve their mental math skills.
Socratic Seminar	A group of students participate in a rigorous, thoughtful dialogue, seeking deeper understanding of complex ideas. Guidelines and language strategies are taught and followed during the seminar.	The teacher presents a distance-versus-time graph and asks students to describe what is happening. Alternatively, the teacher could present an action with four choices of graphs that depict the action. Students choose one of the graphs and explain and defend their choice.
Team Share	Teams take turns to share their final product.	Students work in teams on different math problems. Each team solves its assigned problem cooperatively. The team then has the opportunity to explain its answer with the entire class.
Think–Pair–Share	After the teacher poses a question, students are given time to **think** about their response. The teacher asks students to **pair** up in a specific way (e.g., elbow partners) and **share** their response only with their partner. This strategy helps students practice and refine their response.	What is the difference between prime numbers and composite numbers? Why is it sometimes a good idea to leave an expression in factored form? What is special about the number 1?

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Student Engagement Strategy	Description	Math Example
Think–Write–Pair–Share	This is a variation of think–pair–share. Students are asked to think about their response, write down their response, pair, and share. This strategy might be used when a more complicated response from students is required.	Students are given a word problem to solve. First, they are asked to think about what the problem is asking. Then they are asked to write down their idea. Finally, students share their idea with a partner.
Whiparound	The teacher poses a prompt that has multiple answers. Students write down as many responses as possible. Then the teacher "whips" around the room, calling on one student at a time. Each student shares one of his or her responses. When called on, students should not repeat a response; they must add something new.	What are examples of quadrilaterals? What are some tools you could use to help solve this problem?
Wraparound	After students write their ideas about a topic, each student shares one idea, repeating the statement of the previous student.	The teacher gives the entire class a problem and then allows time for students to write out the steps to solve the problem. Then each student describes one step in the process.

Tools for Mathematics Instruction

There are several instructional tools that teachers can use to make mathematics concepts more concrete for students. A repertoire of tools is especially important in classrooms with English learners or students with disabilities. This section highlights a small number of the tools that teachers can use with their students. (See the Universal Access chapter for more information.)

Visual representations. The MP standards suggest that students look for and make use of structure (**MP.7**), construct viable arguments (**MP.3**), model with mathematics (**MP.4**), and use appropriate tools strategically (**MP.5**). Visual representations can be used to help students achieve proficiency with these standards when used in alignment with the content standards.

To develop student understanding, meaningful relationships between mathematical concepts should be highlighted, not taught in isolation. Diagrams, concept maps, graphic organizers, math drawings, and flowcharts can be used to show relationships (Martinez 2010). Visual representations, such as graphic organizers, combine the use of words and phrases with symbols by using arrows to represent relationships (Burden and Byrd 2010). Ashlock (1998) posits that *concept maps* can be used as an overview of a lesson to summarize what has been taught and to inform instruction. A concept map is a visual organizer in which students place concepts, ideas, and algorithms in bubbles or boxes and connect the bubbles with lines or arrows and a description of how connected bubbles are related. Ashlock notes that these representations are well suited to depict computational procedures and can be created by teachers as well as by students. Visual representations may also be math drawings (e.g., students draw simple pictures to illustrate a story problem) and charts (e.g., fractions and decimals can be sorted and grouped into categories such as greater than one half, equal to one half, and less than one half).

Concrete models. The MP standards advocate the use of concrete models (also known as *manipulatives*) in order for students to make sense of problems and persevere in solving them (**MP.1**) and to use appropriate tools strategically (**MP.5**). Martinez (2010) suggests that learning that utilizes different modes of instruction is necessary to promote both student understanding and recall from long-term memory: "Good teachers know that presenting ideas in a variety of ways can make instruction more effective and more interesting, as well as better able to reach a variety of learners" (Martinez 2010, 229). Concrete models can be utilized to help students learn a wide range of mathematical concepts. For example, students create models to demonstrate the Pythagorean Theorem, they utilize tiles to demonstrate an algebraic expression, and they use base-ten models to demonstrate complex computational procedures.

Interactive technology. New teaching applications for tablet computers and laptops are being created continually. Teachers should feel comfortable about using such technology if it is available to them, but they should view teaching applications and software with a discerning eye to be sure that any technology used in the classroom adheres to the focus, coherence, and rigor of the CCSSM. (See the Technology in the Teaching of Mathematics chapter as well.)

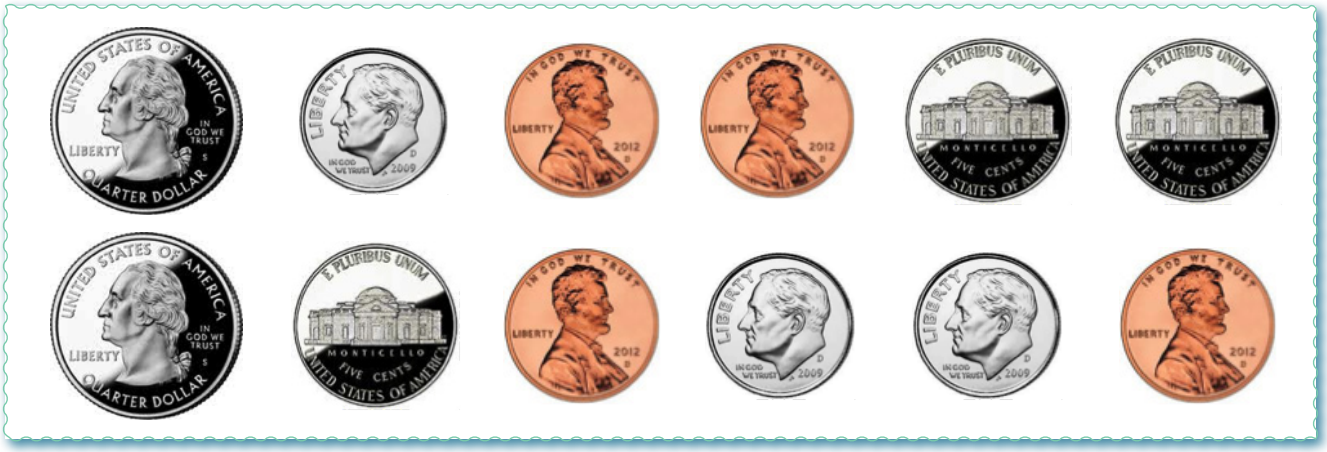
A multitude of instructional resources are available for teachers of mathematics. It would not be possible to name them all in this chapter. Teachers are encouraged to seek multiple sources of information and research to build their instructional repertoire.

Examples of Tasks and Problems Incorporating the MP Standards

The following curricular examples illustrate the types of problems that incorporate the MP standards.

The problem below ("Marissa's Savings") addresses the grade-two standards **2.OA.1** and **2.MD.8**, as well as **MP.1**, **MP.4**, **MP.5**, and **MP.6**. The problem requires students to count a combination of coins and then demonstrate that they understand subtraction of monetary amounts; they do so by writing a story problem that shows how Marissa spends her money.

Marissa's Savings. Marissa has worked very hard to save money, and now she gets to go to the store. How much money does Marissa have? Write a story problem about how Marissa spends her money. Did she have any money left?



This problem demands that students work across a range of mathematical practices. In particular, students practice making sense of problems and persevering in solving them (**MP.1**) by choosing appropriate strategies to use. They apply the mathematics they know to solve problems that arise in everyday life (**MP.4**); utilize available tools, such as concrete models (**MP.5**); and use mathematically precise vocabulary to communicate their explanations by writing a story problem (**MP.6**).

Understanding Perimeter. The following hands-on activity illustrates the grade-three standard **3.MD.8**, as well as **MP.1**, **MP.3**, **MP.5**, and **MP.7**: Students will solve problems with a fixed area and perimeter and develop an understanding of the concept of perimeter by walking around the room, using rubber bands to represent the perimeter of a plane figure on a geoboard, or tracing around a shape on an interactive whiteboard. They find the perimeter of objects, use addition to find perimeters, and recognize the patterns that exist when finding the sum of the lengths and widths of rectangles.

Students use geoboards, tiles, and graph paper to find all the possible rectangles that have a given area (e.g., find the rectangles that have an area of 12 square units). Once students have learned to find the perimeter of a rectangle, they record all the possibilities by using dot or graph paper (**MP.1**); compile the possibilities into an organized list or a table, such as the one shown below (**MP.5**); and determine whether they have all the possible rectangles (**MP.3**). The patterns in the table allow the students to identify the factors of 12, connect the results to the commutative property (**MP.7**), and discuss the differences in perimeter within the same area (**MP.3**). This table can also be used to investigate rectangles with the same perimeter. (It is important to include squares in the investigation.)

Area (square inches)	Length (inches)	Width (inches)	Perimeter (inches)
12	1	12	26
12	2	6	16
12	3	4	14
12	4	3	14
12	6	2	16
12	12	1	26

Source: Kansas Association of Teachers of Mathematics (KATM) 2012, 3rd Grade Flipbook.

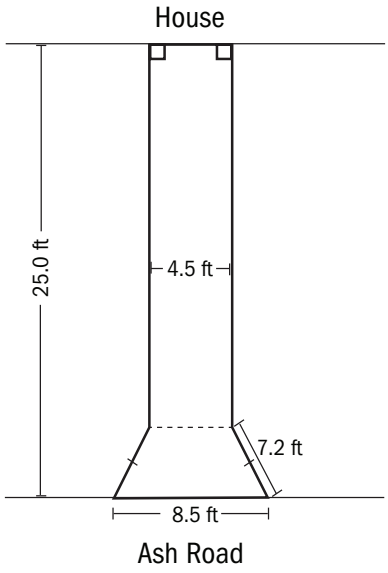
After-School Job. This problem addresses the grade-five standard **5.OA.3**, as well as **MP.1**, **MP.3**, **MP.4**, **MP.5**, and **MP.6**: Leonard needed to earn some money, so he offered to do some extra chores for his mother after school for two weeks. His mother was trying to decide how much to pay him when Leonard suggested the following idea: "You could pay me $1.00 every day for the two weeks, or you can pay me 1¢ for the first day, 2¢ for the second day, 4¢ for the third day, and so on, doubling my pay every day." Which of these two options does Leonard want his mother to choose? Write a letter to Leonard's mother suggesting the option that she should take. Be sure to include drawings that explain your mathematical thinking.

The problem requires students to generate two numerical patterns using two given rules ("add 1" and "double the sum"), generate terms in the resulting sequences over a 14-day period, and explain why the first option would cost Leonard's mother much less money. This problem demands that students

work across a range of mathematical practices. In particular, students practice making sense of problems and persevering in solving them by choosing the strategies to use (**MP.1**). They make conjectures and build a logical progression through careful analyses (**MP.3**); apply the mathematics they know to solve problems of interest to them that arise in everyday life (**MP.4**); utilize available tools, such as concrete models and calculators (**MP.5**); and use mathematically precise vocabulary to communicate their explanations through writing and through graphics, such as charts (**MP.6**).

The following problem ("Ms. Olsen's Sidewalk" [Smarter Balanced Assessment Consortium 2011, 111]) addresses the grade-seven standards **7.G.6** and **7.NS.3**, the grade-eight standard **8.G.7**, and MP standards **MP.1**, **MP.4**, and **MP.6**. In this task, students are given a real-world problem whose solution involves determining the areas of two-dimensional shapes as part of calculating the cost of a sidewalk.

Ms. Olsen's Sidewalk. Ms. Olsen is having a new house built on Ash Road. She is designing a sidewalk from Ash Road to her front door. Ms. Olsen wants the sidewalk to end in the shape of an isosceles trapezoid, as shown in this diagram:



The contractor charges a fee of $200 plus $12 per square foot of sidewalk. Based on the diagram, what will the contractor charge Ms. Olsen for her sidewalk? Show your work or explain how you found your answer.

A common problem in calculating the area of a trapezoid is the misuse of the length marked 7.2 feet. Students need to make use of this dimension, but they must avoid multiplying 8.5×7.2 in an attempt to find the area of the trapezoid. Once the decision has been made regarding how to best deconstruct the figure, the students need to apply the Pythagorean Theorem to calculate the length of the path connected to the trapezoid.

When this has been calculated, the remaining length and area calculations can be undertaken. The final stage of this multi-step problem is to calculate the cost of the paving based on the basic fee of $200 plus $12 per square foot. This task demands that students work across a range of mathematical practices. In particular, they need to make sense of the problem and persevere in solving it (**MP.1**), analyze the information given, and choose a solution pathway.

Furthermore, students need to attend to precision (**MP.6**) in their careful use of units in the cost calculations. In providing a written rationale of their work, both English learners and native speakers may experience linguistic difficulties in formulating their positions. Additional assistance from the teacher may be required.

The problem below ("Baseball Jerseys") comes from the Mathematics Assessment Resource Service (MARS) Web site (http://map.mathshell.org/stds.php [accessed September 1, 2015]). It addresses the grade-seven standards **7.EE.4** and **7.NS.3**, the grade-eight standards **8.EE.8** and **8.F.4**, and MP standards **MP.1**, **MP.4**, and **MP.7**.

Baseball Jerseys. Bill is going to order new jerseys for his baseball team. The jerseys will have the team logo printed on the front. Bill asks two local companies to give him a price for producing the jerseys. The first company, Print It, will charge $21.50 for each jersey. The second company, Top Print, has a setup cost of $70 and then charges $18 for each jersey. Figure out how many jerseys Bill would need to order so that the cost for Top Print would be less than the cost for Print It. Explain your answer.

Students may utilize the following approaches to solve this problem:

a. Using n for the number of jerseys ordered and c for the total cost in dollars, write an equation to show the total cost of jerseys from Print It.

b. Using n to stand for the number of jerseys ordered and c for the total cost in dollars, write an equation to show the total cost of jerseys from Top Print.

c. Use the two equations from questions (a) and (b) to figure out how many jerseys Bill would need to order so that the total cost for Top Print would be less than the total cost for Print It.

This problem considers the costing models of two print companies, and students should be able to produce two equations: $c = 21.5n$ and $c = 70 + 18n$. The third part of this task may be a bit more challenging. Students may construct the inequality $70 + 18n < 21.5n$ and then solve for n.

This problem also demands that students work across a range of mathematical practices. In particular, students practice making sense of problems and persevering in solving them (**MP.1**) by choosing what strategies to use. They also look for and make use of structure (**MP.7**) in that understanding the properties of linear growth leads to a solution for the problem. Finally, students practice modeling (**MP.4**) as they construct equations.

Several Internet resources provide grade-level curricular examples that are aligned with the CA CCSSM (including the MP standards). These include Department of Education Web sites from other states that have adopted the Common Core State Standards. References to these resources can be found through-out this framework. The MARS Web site (http://map.mathshell.org/stds.php) provides a multitude of exercises that focus specifically on the MP standards.

Real-World Problems

Teachers do not use real-world situations to serve mathematics; they use mathematics to serve and address real-world situations. Real-world problems provide opportunities for mathematics to be learned and engaged in context. Miller (2011) cautions that when students are assigned the task of performing real-world mathematics, the CA CCSSM do not simply want students to mimic real-world connections; the intent is for students to be able to successfully solve related mathematics problems. Students are already conditioned to do tasks. Even when a task might have strong connections to the real world, it can still be just that: a task to complete. Teachers need to keep this in mind when they ask students to perform real-world mathematics, just as the CA CCSSM suggest (Miller 2011).

In "Exploring World Maps," adapted from the California Mathematics Project (2012), students work toward mastery of standard **6.RP.3**, which calls for the use of ratio and rate reasoning to solve real-world and mathematical examples. Students are provided with a world map and are given Mexico's surface area (750,000 square miles). Then students are asked to use this information and other available tools, such as tracing paper and centimeter grids (**MP.5**), to estimate areas of several countries and continents. Finally, students are asked to provide short answers to the following questions:

a. Which area did you estimate to be larger—Mexico or Alaska?

b. Approximately how many times can Greenland fit into Africa?

c. Do you feel confident in your estimations?

d. What estimation methods did you use?

e. Now that you know the actual areas (students are provided with the actual areas prior to answering this question), what surprised you the most?

f. How does the location of the equator affect how this map is viewed?

Once again, teachers should be cognizant of potential linguistic difficulties that may affect English learners and native speakers alike. Schleppegrell (2007) notes that counting, measuring, and other "everyday" ways of doing mathematics draw on everyday language, but that the kind of mathematics that students need to develop through schooling uses language in new ways to serve new functions. It is the teacher's job to assist all students in acquiring this new language.